

DUPIN HYPERSURFACES IN LIE SPHERE GEOMETRY

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INTRODUCTION

The method of moving frames in Lie sphere geometry has produced significant results in the classification of Dupin hypersurfaces in spheres, as seen, for example in the papers by Pinkall [Pin85], Cecil and Chern [CC89], Cecil and Jensen [CJ98, CJ00], Cecil, Chi, and Jensen [CCJ07]. What is the secret of its effectiveness? The answer emerges in the classification of nonumbilic isoparametric surfaces in the space form geometries. Using the method of moving frames, the proof of this classification is an elementary exercise. The same proof classifies the cyclides of Dupin in Möbius geometry and finally in Lie sphere geometry, where all nonumbilic Dupin immersions are Lie sphere congruent to each other. The idea of this proof extends to the cases of higher dimensions and greater number of principal curvatures. For more details and examples see the forthcoming book by the author, Musso, and Nicolodi [JMN].

1. METHOD OF MOVING FRAMES

The method of moving frames generally refers to use of the principal bundle of linear frames over a manifold N . The text books [KN63, KN69] by Shoshichi Kobayashi and Katsumi Nomizu, contain a seminal exposition of this general method. In this paper, the method of moving frames refers to the more specialized case of the linear frames coming from a Lie group G acting transitively on a manifold N . In this case, if G_0 denotes the isotropy subgroup of G at a chosen origin $o \in N$, then

$$\pi : G \rightarrow N, \quad \pi(g) = g(o)$$

is the projection map of a principal G_0 -bundle over N . A moving frame in N is any local section of this bundle. A moving frame along an immersed submanifold $f : M \rightarrow N$ is a smooth map $e : U \subset M \rightarrow G$ such that $f = \pi \circ e$. In general, there are many local frames along f . If $e : U \rightarrow G$ is a frame along f , then for any smooth map $K : U \rightarrow G_0$, the map $eK : U \rightarrow G$ is also a frame along f . The method of moving

frames develops a process for reducing the local frames along f to a *best frame*.

The Maurer-Cartan form of G is the left-invariant \mathfrak{g} -valued 1-form $\omega = g^{-1}dg$ on G . If G is a matrix group, say G is a closed subgroup of the general linear group $\mathbf{GL}(n, \mathbf{R})$, then $\omega = (\omega_j^i) \in \mathfrak{gl}(n, \mathbf{R})$ is an $n \times n$ matrix of left invariant 1-forms on G that satisfy the *structure equations* of G , $d\omega = -\omega \wedge \omega$, which in components is

$$d\omega_j^i = - \sum_k \omega_k^i \wedge \omega_j^k.$$

Frame reduction is a systematic way of imposing linear relations on the forms ω_j^i of $e^*\omega$, for a frame $e : U \rightarrow G$ along f .

The following two Cartan-Darboux theorems provide the basic analytic tools in the method of moving frames.

Congruence Theorem. *If $e, \tilde{e} : M \rightarrow G$ are smooth maps from a connected manifold M such that $e^*\omega = \tilde{e}^*\omega$, then there exists an element $g \in G$ such that $\tilde{e} = ge$ on M .*

Existence Theorem. *If η is a \mathfrak{g} -valued 1-form on a simply connected manifold M such that $d\eta = -\eta \wedge \eta$, then there exists a smooth map $e : M \rightarrow G$ such that $\eta = e^*\omega$.*

2. EUCLIDEAN SPACE

Consider an immersion $\mathbf{x} : M \rightarrow \mathbf{R}^3$, where M is always now a connected surface. Our group now is the Euclidean group $\mathbf{E}(3) = \mathbf{R}^3 \rtimes \mathbf{SO}(3)$, which is represented as a matrix group by $(\mathbf{y}, A) = \begin{pmatrix} 1 & 0 \\ \mathbf{y} & A \end{pmatrix} \in \mathbf{GL}(4, \mathbf{R})$. It acts transitively by $(\mathbf{y}, A)\mathbf{x} = \mathbf{y} + A\mathbf{x}$. The principal $\mathbf{SO}(3)$ -bundle $\pi : \mathbf{E}(3) \rightarrow \mathbf{R}^3$, given by $(\mathbf{y}, A)\mathbf{0} = \mathbf{y}$, is the bundle of all oriented orthonormal frames on \mathbf{R}^3 . A frame along \mathbf{x} is a smooth map $(\mathbf{x}, e) : U \subset M \rightarrow \mathbf{E}(3)$. If \mathbf{e}_i denotes column i of $e \in \mathbf{SO}(3)$, then $e = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal frame at each point of \mathbf{x} . The pull-back of the Maurer-Cartan form $(\mathbf{x}, e)^{-1}d(\mathbf{x}, e) = (\theta, \omega)$ satisfies

$$d\mathbf{x} = \sum_1^3 \theta^i \mathbf{e}_i, \quad d\mathbf{e}_i = \sum_1^3 \omega_i^j \mathbf{e}_j,$$

where $\theta = (\theta^i)$ and $\omega = (\omega_j^i) = -{}^t\omega$. The structure equations are

$$d\theta^i = - \sum_1^3 \omega_j^i \wedge \theta^j, \quad d\omega_j^i = - \sum_1^3 \omega_k^i \wedge \omega_j^k.$$

The best frame along \mathbf{x} is one for which \mathbf{e}_3 is normal to \mathbf{x} and \mathbf{e}_1 and \mathbf{e}_2 are principal directions. The pull-back of the Maurer-Cartan form, $(\mathbf{x}, e)^{-1}d(\mathbf{x}, e) = (\theta, \omega)$, expresses these conditions by

$$\theta^3 = 0, \quad \omega_1^3 = a\theta^1, \quad \omega_2^3 = c\theta^2,$$

where the functions $a, c : U \rightarrow \mathbf{R}$ are the principal curvatures. The immersion is totally umbilic if $a = c$ at every point of M , in which case a is constant on M , and $\mathbf{x}(M)$ lies in a plane, if $a = 0$ or $\mathbf{x}(M)$ lies in a sphere of radius $1/|a|$, if $a \neq 0$.

Definition 1. The immersion $\mathbf{x} : M \rightarrow \mathbf{R}^3$ is *isoparametric* if its principal curvatures are constant on M . The immersion is *Dupin* if its principal curvatures are distinct and each is constant along its own lines of curvature.

Nonumbilic isoparametric immersions are Dupin. The proof of the following elementary theorem for nonumbilic isoparametric immersions is the prototype of the classification of Dupin hypersurfaces in spheres.

Theorem 2. *If $\mathbf{x} : M \rightarrow \mathbf{R}^3$ is isoparametric with distinct principal curvatures a and c , then $ac = 0$, and $\mathbf{x}(M)$ lies in a circular cylinder of radius R , where $1/R$ is the absolute value of the nonzero principal curvature.*

Proof. For the frame $(\mathbf{x}, e) : U \rightarrow \mathbf{E}(3)$ above,

$$\theta^3 = 0, \quad \omega_1^3 = a\theta^1, \quad \omega_2^3 = c\theta^2,$$

and $\theta^1 \wedge \theta^2$ is never zero, since \mathbf{x} is an immersion. Then a and c constant and distinct combined with the structure equations, implies $ac = 0$ and $\omega_2^1 = 0$. Suppose $a \neq 0$. Now the pull back of the Maurer-Cartan form satisfies

$$\theta^3 = 0, \quad \omega_2^1 = 0, \quad \omega_1^3 = a\theta^1, \quad \omega_2^3 = 0.$$

These equations on the components of the Maurer-Cartan form itself define a completely integrable, left-invariant, 2-plane distribution \mathfrak{h} on $\mathbf{E}(3)$, so \mathfrak{h} is a Lie subalgebra of $\mathcal{E}(3)$. If H is the connected Lie subgroup whose Lie algebra is \mathfrak{h} , then the maximal integral surfaces of this distribution are the right cosets of H . Any frame of the above type along \mathbf{x} is an integral surface of \mathfrak{h} , so it lies in a right coset $(\mathbf{y}, A)H$, and the projection $\mathbf{x}(U) = (\mathbf{y}, A)H\mathbf{0}$. M connected implies all of $\mathbf{x}(M)$ lies in this projection. But this projection is congruent to $H\mathbf{0}$, which by exponentiation of \mathfrak{h} is the cylinder $x^2 + (z - \frac{1}{a})^2 = \frac{1}{a^2}$ in \mathbf{R}^3 . \square

Relative to the best frame $(\mathbf{x}, e) : U \rightarrow \mathbf{E}(3)$ above, $da = a_1\theta^1 + a_2\theta^2$ and the lines of curvature of a are the integral curves of $\theta^2 = 0$, so a is constant along its lines of curvature iff $a_1 = 0$. Similarly, $c_2 = 0$ is

the condition for c to be constant along its lines of curvature. A Dupin immersion is a solution of the system of PDE

$$a_2 = p(a - c), \quad c_1 = q(a - c), \quad p_2 - q_1 = ac + p^2 + q^2,$$

where $da = a_2\theta^2$, $dc = c_1\theta^1$, and $\omega_1^2 = p\theta^1 + q\theta^2$. The situation is essentially the same in the three space form geometries.

3. SPACE FORM GEOMETRIES

The space form geometries are

- Euclidean space $\mathbf{R}^3 = \mathbf{E}(3)/\mathbf{SO}(3)$.
- Spherical geometry $\mathbf{S}^3 = \mathbf{SO}(4)/\mathbf{SO}(3) = \{\mathbf{x} = \sum_0^3 x^i \epsilon_i \in \mathbf{R}^4 : \mathbf{x} \cdot \mathbf{x} = \sum_0^3 (x^i)^2 = 1\}$.
- Hyperbolic geometry $\mathbf{H}^3 = \mathbf{SO}(3, 1)/\mathbf{SO}(3) = \{\mathbf{x} = \sum_1^4 x^i \epsilon_i \in \mathbf{R}^{3,1} : \langle \mathbf{x}, \mathbf{x} \rangle = \sum_1^3 (x^i)^2 - (x^4)^2 = 0\}$.

Let $\text{span}\{\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5\} = \mathbf{R}^{4,2}$ for the standard orthonormal basis of signature $++++--$, so

$$\begin{aligned} \mathbf{R}^3 &= \text{span}\{\epsilon_1, \epsilon_2, \epsilon_3\}, & \mathbf{R}^4 &= \text{span}\{\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3\}, \\ \mathbf{R}^{3,1} &= \text{span}\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}, & \mathbf{R}^{4,1} &= \text{span}\{\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}. \end{aligned}$$

Theorem 3. *The nonumbilic isoparametric immersions in \mathbf{S}^3 have constant principal curvatures $a \neq c$ satisfying $ac + 1 = 0$. They are the 1-parameter family of circular tori $\mathbf{S}^1(r) \times \mathbf{S}^1(s) \subset \mathbf{S}^3$, where $r = \cos \alpha$, $s = \sin \alpha$, and the parameter is $0 < \alpha \leq \pi/4$. Here $\mathbf{S}^1(r)$ is the circle in \mathbf{R}^2 with center at the origin and radius r , and $a = -\tan \alpha$, $c = \cot \alpha$.*

Theorem 4. *The nonumbilic isoparametric immersions in \mathbf{H}^3 have constant principal curvatures $a \neq c$ satisfying $ac - 1 = 0$. They are the 1-parameter family of circular hyperboloids $\mathbf{S}^1(\frac{a}{b}) \times \mathbf{H}^1(\frac{1}{b}) \in \mathbf{H}^3$, where the parameter is $0 < a < 1$ and $b = \sqrt{1 - a^2}$. Here $\mathbf{S}^1(\frac{a}{b}) \subset \mathbf{R}^2 = \text{span}\{\epsilon_1, \epsilon_2\}$, and $\mathbf{H}^1(\frac{1}{b}) \subset \mathbf{R}^{1,1} = \text{span}\{\epsilon_3, \epsilon_4\}$ is the hyperboloid $z^2 - w^2 = -b^2$.*

The moving frame proofs in both cases are essentially identical to that for the Euclidean case. Dupin immersions into \mathbf{S}^3 and \mathbf{H}^3 are defined in the same way as in the Euclidean case.

The space form geometries are related by conformal diffeomorphisms that send spheres to spheres. These are, stereographic projection

$$\mathcal{S} : \mathbf{S}^3 \setminus \{-\epsilon_0\} \rightarrow \mathbf{R}^3, \quad \mathcal{S}\left(\sum_0^3 x^i \epsilon_i\right) = \frac{\sum_1^3 x^i \epsilon_i}{1 + x^0},$$

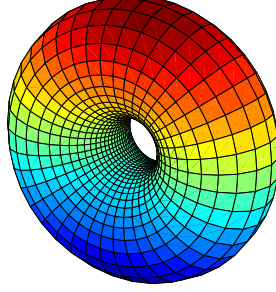


FIGURE 1. Stereographic projection of circular torus with $\alpha = \pi/4$.

with inverse

$$\mathcal{S}^{-1}\left(\sum_1^3 y^i \epsilon_i\right) = \frac{(1 - \sum_1^3 (y^i)^2) \epsilon_0 + 2 \sum_1^3 y^i \epsilon_i}{1 + \sum_1^3 (y^i)^2}.$$

Stereographic projection of the circular torus with parameter $\alpha = \pi/4$ is shown in Figure 1. It is a Dupin immersion, but not isoparametric, into \mathbf{R}^3 .

Hyperbolic stereographic projection onto the unit ball $\mathbf{B}^3 \subset \mathbf{R}^3$ is

$$\mathfrak{s} : \mathbf{H}^3 \rightarrow \mathbf{B}^3, \quad \mathfrak{s}\left(\sum_1^4 x^i \epsilon_i\right) = \frac{\sum_1^3 x^i \epsilon_i}{1 + \mathbf{x}^4},$$

with inverse

$$\mathfrak{s}^{-1}\left(\sum_1^3 y^i \epsilon_i\right) = \frac{2 \sum_1^3 y^i \epsilon_i + (1 + \sum_1^3 (y^i)^2) \epsilon_4}{1 - \sum_1^3 (y^i)^2}.$$

It is an isometry onto the ball with the Poincaré metric $I_{\mathbf{B}} = \frac{4d\mathbf{y} \cdot d\mathbf{y}}{(1 - |\mathbf{y}|^2)^2}$, and it is a conformal embedding when regarded as a map into \mathbf{R}^3 by the conformal inclusion $\mathbf{B}^3 \subset \mathbf{R}^3$. Then $\mathcal{S}^{-1} \circ \mathfrak{s} : \mathbf{H}^3 \rightarrow \mathbf{S}^3$ is a conformal embedding. The projection by \mathfrak{s} of the circular hyperboloid with parameter $a = 1/2$ is shown in Figure 2. It is an isoparametric immersion into the Poincaré ball. It is Dupin, but not isoparametric, into \mathbf{R}^3 .

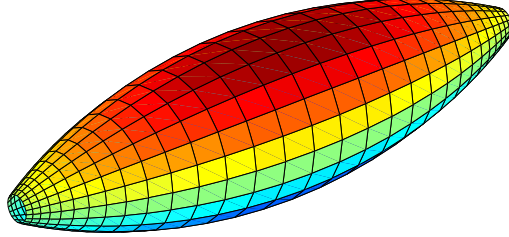


FIGURE 2. Hyperbolic stereographic projection of circular hyperboloid with $a = 1/2$.

4. TANGENT SPHERES

These conformal diffeomorphisms do not send isoparametric immersions to isoparametric immersions, but they do send Dupin immersions to Dupin immersions. This is easily seen if we reformulate the Dupin condition in terms of tangent sphere maps. The reformulated definition makes sense in Möbius and Lie sphere geometries, as well as in the space forms.

Definition 5. The *oriented sphere of center $\mathbf{m} \in \mathbf{S}^3$ and signed radius r* , where $(0 < r < \pi)$, is

$$S_r(\mathbf{m}) = \{\mathbf{x} \in \mathbf{S}^3 : \mathbf{x} \cdot \mathbf{m} = \cos r\}$$

oriented by the unit normal $\mathbf{n}(\mathbf{x}) = \frac{\mathbf{m} - \cos r \mathbf{x}}{\sin r}$. Note that $S_r(-\mathbf{m})$ is $S_{\pi-r}(\mathbf{m})$ with the opposite orientation.

The set of all oriented spheres in \mathbf{S}^3 is identified with the smooth hypersurface

$$\mathbf{S}^{3,1} = \{S = \sum_0^4 s^i \epsilon_i \in \mathbf{R}^{4,1} : \langle S, S \rangle = \sum_0^3 (s^i)^2 - (s^4)^2 = 1\},$$

by $S_r(\mathbf{m}) \leftrightarrow \frac{\mathbf{m} + \cos r \epsilon_4}{\sin r}$ and $S = \sum_0^4 s^i \epsilon_i \leftrightarrow S_r(\frac{\sum_0^3 s^i \epsilon_i}{\sin r})$, where $\cot r = \frac{s^4}{s^3}$.

Let $\mathbf{x} : M \rightarrow \mathbf{S}^3$ be an immersed surface with unit normal vector field \mathbf{e}_3 along \mathbf{x} .

Definition 6. An *oriented tangent sphere* to \mathbf{x} at a point is an oriented sphere tangent to the surface at \mathbf{x} with unit normal at \mathbf{x} equal to \mathbf{e}_3 . It is one of the spheres $S_r(\cos r \mathbf{x} + \sin r \mathbf{e}_3)$, where $0 < r < \pi$. A tangent sphere is a *curvature sphere* to \mathbf{x} if $\cot r$ is a principal curvatures of \mathbf{x} at the point.

Definition 7. A *tangent sphere map along* $\mathbf{x} : M \rightarrow \mathbf{S}^3$ is a smooth map $S : M \rightarrow \mathbf{S}^{3,1}$ such that $S(m)$ is an oriented tangent sphere to \mathbf{x} at $\mathbf{x}(m)$, for all $m \in M$. It must be given by

$$S = \frac{\cos r (\mathbf{x} + \epsilon_4) + \sin r \mathbf{e}_3}{\sin r} = \cot r (\mathbf{x} + \epsilon_4) + \mathbf{e}_3.$$

for smooth function $r : M \rightarrow (0, \pi)$.

If $S : M \rightarrow \mathbf{S}^{3,1}$ is a tangent sphere map along \mathbf{x} , then $dS = ((\cot r - a)\mathbf{e}_1 + (\cot r)_1(\mathbf{x} + \epsilon_4))\omega_0^1 + ((\cot r - c)\mathbf{e}_2 + (\cot r)_2(\mathbf{x} + \epsilon_4))\omega_0^2$, where $d \cot r = (\cot r)_1\omega_0^1 + (\cot r)_2\omega_0^2$. Thus, $dS_m \bmod (\mathbf{x} + \epsilon_4)$ has rank less than 2 if and only if $\cot r$ is a principal curvature of \mathbf{x} at m if and only if $S(m)$ is a curvature sphere of \mathbf{x} . If S is a curvature sphere map along \mathbf{x} , then dS has rank less than 2 at every point if and only if $\cot r$ is constant along its lines of curvature. This gives us an alternate, but equivalent, definition of Dupin immersion into \mathbf{S}^3 .

Definition 8. An immersion $\mathbf{x} : M \rightarrow \mathbf{S}^3$ is Dupin if dS is singular at every point of M , for any curvature sphere map $S : M \rightarrow \mathbf{S}^{3,1}$ along \mathbf{x} .

5. MÖBIUS GEOMETRY

Möbius space is \mathbf{S}^3 acted upon by the group of all of its conformal diffeomorphisms. With the projective description

$$\mathcal{M} = \{[q] = [\sum_0^4 q^i \epsilon_i] \in P(\mathbf{R}^{4,1}) : \sum_0^3 (q^i)^2 - (q^4)^2 = \langle q, q \rangle = 0\},$$

we have a conformal diffeomorphism

$$f_+ : \mathbf{S}^3 \rightarrow \mathcal{M}, \quad f_+(\sum_0^3 x^i \epsilon_i) = [\sum_0^3 x^i \epsilon_i + \epsilon_4],$$

and the group of all conformal transformations on \mathbf{S}^3 is represented on \mathcal{M} as the group of linear transformations $\mathbf{SO}(\mathbf{R}^{4,1})$. Let $\mathbf{Möb} \subset \mathbf{GL}(5, \mathbf{R})$ be the matrix representation of this group in the basis

$$\delta_0 = \frac{\epsilon_4 + \epsilon_0}{\sqrt{2}}, \quad \delta_i = \epsilon_i, \quad \delta_4 = \frac{\epsilon_4 - \epsilon_0}{\sqrt{2}}$$

of $\mathbf{R}^{4,1}$. Then $\mathcal{M} = \mathbf{Möb}/G_0$, where G_0 is the isotropy subgroup at the origin $[\delta_0] = f_+(\epsilon_0)$. The columns \mathbf{Y}_a of an element $Y \in \mathbf{Möb}$ form a Möbius frame of $\mathbf{R}^{4,1}$:

$$(\langle \mathbf{Y}_a, \mathbf{Y}_b \rangle)_{0 \leq a, b \leq 4} = g = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_3 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The space forms are conformally contained in \mathcal{M} by

$$f_+ : \mathbf{S}^3 \rightarrow \mathcal{M}, \quad f_0 = f_+ \circ \mathcal{S}^{-1} : \mathbf{R}^3 \rightarrow \mathcal{M}, \quad f_- = f_0 \circ \mathfrak{s} : \mathbf{H}^3 \rightarrow \mathcal{M}.$$

These embeddings are equivariant with natural monomorphisms

$$F_+ : \mathbf{SO}(4) \rightarrow \mathbf{Möb}, \quad F_0 : \mathbf{E}(3) \rightarrow \mathbf{Möb}, \quad F_- : \mathbf{SO}(3, 1) \rightarrow \mathbf{Möb}.$$

A Möbius frame along an immersed surface $f : M \rightarrow \mathcal{M}$ is a smooth map $Y : U \subset M \rightarrow \mathbf{Möb}$ such that $f = [\mathbf{Y}_0]$ on U . Then $d\mathbf{Y}_b = \sum_0^4 \omega_b^a \mathbf{Y}_a$ and $\omega_0^4 = -\langle d\mathbf{Y}_0, \mathbf{Y}_0 \rangle = 0$, since $\langle \mathbf{Y}_0, \mathbf{Y}_0 \rangle = 0$ is constant on U . It is a first order frame if $\omega_0^3 = 0$ on U . That is, $d\mathbf{Y}_0 = \sum_0^2 \omega_0^a \mathbf{Y}_a$. For such a frame, $\mathbf{Y}_3 : U \rightarrow \mathbf{S}^{3,1}$ is a tangent sphere map along f , since $\langle \mathbf{Y}_0, \mathbf{Y}_3 \rangle = 0$ implies that $[\mathbf{Y}_0]$ lies on this sphere, and $\langle d\mathbf{Y}_0, \mathbf{Y}_3 \rangle = 0$ implies that the sphere is tangent to f at each point. Moreover, for any smooth function $r : U \rightarrow \mathbf{R}$, the map $\mathbf{Y}_3 + r\mathbf{Y}_0 : U \rightarrow \mathbf{S}^{3,1}$ is a tangent sphere map along f . This is a curvature sphere map iff

$$d(\mathbf{Y}_3 + r\mathbf{Y}_0) \equiv \sum_1^2 (\omega_3^a + r\omega_0^a) \mathbf{Y}_a \pmod{\mathbf{Y}_0}$$

has rank less than 2 at each point of U , which is equivalent to

$$(\omega_3^1 + r\omega_0^1) \wedge (\omega_3^2 + r\omega_0^2) = 0$$

at every point of U . If we assume distinct curvature spheres at each point of M , then about each point we can reduce to a best frame $Y : U \rightarrow \mathbf{Möb}$, characterized by

$$\begin{aligned} \omega_0^3 &= 0, \quad \omega_0^1 \wedge \omega_0^2 > 0 && \text{(first order)} \\ \omega_1^3 - i\omega_2^3 &= \omega_0^1 + i\omega_0^2 && \text{(second order)} \\ \omega_3^0 &= 0 && \text{(third order).} \end{aligned}$$

The structure equations then imply

$$\begin{aligned} \omega_1^2 &= q_1\omega_0^1 + q_2\omega_0^2, & \omega_0^0 &= -2(q_2\omega_0^1 - q_1\omega_0^2), \\ \omega_1^0 &= p_1\omega_0^1 + p_2\omega_0^2, & \omega_2^0 &= -p_2\omega_0^1 + p_3\omega_0^2, \end{aligned}$$

for smooth functions $q_1, q_2, p_1, p_2, p_3 : U \rightarrow \mathbf{R}$, which satisfy

$$d(q_2 + iq_1) \wedge \varphi = -\frac{1}{2}(p_1 + p_3 + 1 + q_1^2 + q_2^2 + ip_2)\varphi \wedge \bar{\varphi},$$

$$d(p_1 + p_3 - i2p_2) \wedge \varphi + d(p_1 - p_3) \wedge \bar{\varphi} = (2p_2 + i(p_1 + p_3))(q_1 + iq_2)\varphi \wedge \varphi,$$

where $\varphi = \omega_0^1 + i\omega_0^2$. Relative to a best frame $Y : U \rightarrow \mathbf{Möb}$, the curvature sphere maps are $\mathbf{Y}_3 + \epsilon\mathbf{Y}_0$, where $\epsilon = \pm 1$. The Dupin condition is that these maps are singular at every point. But

$$d(\mathbf{Y}_3 + \epsilon\mathbf{Y}_0) = (\epsilon - 1)\omega_0^1\mathbf{Y}_1 + (\epsilon + 1)\omega_0^2\mathbf{Y}_2 + \epsilon\omega_0^0\mathbf{Y}_0$$

is singular at each point, for both choices of $\epsilon = \pm 1$ iff $\omega_0^0 = 0$ iff $q_1 = q_2 = 0$ on U . From the structure equations it then follows that $f : M \rightarrow \mathcal{M}$ is Dupin iff $p_2 = 0$ as well,

$$\omega_0^0 = 0 = \omega_1^2, \quad \omega_1^0 = p_1 \omega_0^1, \quad \omega_2^0 = p_3 \omega_0^2,$$

and

$$p_1 + p_3 = -1, \quad p_1 - p_3 = 2C,$$

for some constant $C \in \mathbf{R}$. Hence, a best frame field $Y : U \rightarrow \mathbf{Möb}$ along a Dupin immersion is an integral surface of the completely integrable, left-invariant 2-plane distribution on $\mathbf{Möb}$ defined, for each $C \in \mathbf{R}$, by the Lie subalgebra $\mathfrak{h}_C \subset \mathfrak{m\ddot{o}b}$ defined by the equations

$$\begin{aligned} \omega_0^0 &= 0, & \omega_1^0 &= \left(-\frac{1}{2} + C\right)\omega_0^1, & \omega_2^0 &= \left(-\frac{1}{2} - C\right)\omega_0^2, & \omega_3^0 &= 0, \\ \omega_1^2 &= 0, & \omega_1^3 &= \omega_0^1, & \omega_2^3 &= -\omega_0^2, & \omega_0^3 &= 0. \end{aligned}$$

If $H_C \subset \mathbf{Möb}$ is the connected Lie subgroup of \mathfrak{h}_C , then the maximal integral submanifolds of this distribution are the right cosets of H_C . The projection $H_C[\delta_0]$ is a Dupin surface in \mathcal{M} , with $\pm C$ essentially the same. These are:

- f_+ of circular tori of parameter $0 < \alpha \leq \pi/4$ for $0 \leq C = \cos 2\alpha < 1$.
- f_0 of the circular cylinder of radius 1 for $C = 1$.
- f_- of the circular hyperboloids of parameter $0 < a < 1$ for $C = \frac{a^{-1}+a}{a^{-1}-a} > 1$.

We have thus proved that any Dupin immersion of a connected surface into Euclidean space is contained in a conformal transformation of a nonumbilic isoparametric immersion into some space form.

6. LIE SPHERE GEOMETRY

The fundamental role of tangent sphere maps along an immersion $f : M \rightarrow \mathcal{M}$ provides motivation to linearize the set of all oriented tangent spheres at a point of f . We call such a set a *pencil of oriented spheres* at a point of f . Projectivize $\mathbf{S}^{3,1}$ by the *Lie quadric*

$$Q = \{[q] \in P(\mathbf{R}^{4,2}) : \langle q, q \rangle = 0\},$$

by the inclusion $\mathbf{S}^{3,1} \subset Q$ given by $S \mapsto S + \epsilon_5$. Möbius space itself is a subset of Q by the natural inclusion $\mathbf{R}^{4,1} \subset \mathbf{R}^{4,2}$. Then

$$Q = \mathcal{M} \cup \mathbf{S}^{3,1} = \{[q] : \langle q, \epsilon_5 \rangle = 0\} \cup \{[q] : \langle q, \epsilon_5 \rangle \neq 0\}$$

is a disjoint union comprising all oriented spheres in \mathcal{M} , including the point spheres \mathcal{M} (radius is zero), which have no orientation. A pencil

of oriented spheres is a line in Q , which is the linear span of any two orthogonal elements $[S_0], [S_1] \in Q$, denoted $[S_0, S_1]$.

Definition 9. The space of Lie sphere geometry is Λ , the set of all lines in Q .

Λ is a smooth manifold of dimension five. Any point $\lambda \in \Lambda$ contains a unique point sphere, which then defines the *spherical projection map*

$$\sigma : \Lambda \rightarrow \mathcal{M}, \quad \sigma(\lambda) = \lambda \cap \mathcal{M},$$

which is a fiber bundle with standard fiber equal to \mathbf{S}^2 . The group $\mathbf{SO}(\mathbf{R}^{4,2})$ preserves Q and acts transitively on Λ . As origin of Λ choose

$$o = [\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1], \quad \boldsymbol{\lambda}_0 = \frac{\boldsymbol{\epsilon}_5 + \boldsymbol{\epsilon}_0}{\sqrt{2}}, \quad \boldsymbol{\lambda}_1 = \frac{\boldsymbol{\epsilon}_4 + \boldsymbol{\epsilon}_1}{\sqrt{2}} \in \mathbf{R}^{4,2}.$$

A *Lie frame* of $\mathbf{R}^{4,2}$ is a basis $\mathbf{T}_0, \dots, \mathbf{T}_5$ for which

$$(\langle \mathbf{T}_a, \mathbf{T}_b \rangle)_{0 \leq a, b \leq 5} = \hat{g} = \begin{pmatrix} 0 & 0 & -L \\ 0 & I_2 & 0 \\ -L & 0 & 0 \end{pmatrix},$$

where $L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The *Lie sphere group* is

$$G = \{T \in \mathbf{GL}(6, \mathbf{R}) : {}^t T \hat{g} T = \hat{g}\},$$

the representation of $\mathbf{SO}(\mathbf{R}^{4,2})$ in the Lie frame $\boldsymbol{\lambda}_0, \dots, \boldsymbol{\lambda}_5$, where

$$\boldsymbol{\lambda}_2 = \boldsymbol{\epsilon}_2, \quad \boldsymbol{\lambda}_3 = \boldsymbol{\epsilon}_3, \quad \boldsymbol{\lambda}_4 = \frac{\boldsymbol{\epsilon}_4 - \boldsymbol{\epsilon}_1}{\sqrt{2}}, \quad \boldsymbol{\lambda}_5 = \frac{\boldsymbol{\epsilon}_5 - \boldsymbol{\epsilon}_0}{\sqrt{2}}.$$

Then $T \in G$ iff the columns $\mathbf{T}_0, \dots, \mathbf{T}_5$ of T form a Lie frame. Its Lie algebra in this representation is

$$\mathfrak{g} = \{\mathcal{T} \in \mathfrak{gl}(6, \mathbf{R}) : {}^t \mathcal{T} \hat{g} + \hat{g} \mathcal{T} = 0\}.$$

The Maurer-Cartan form of G is $\omega = (\omega_b^a) \in \mathfrak{g}$. If G_0 is the isotropy subgroup of G at o , then the projection

$$\pi : G \rightarrow \Lambda, \quad \pi(T) = [\mathbf{T}_0, \mathbf{T}_1]$$

is a principal G_0 -bundle. A contact structure is given on Λ by the 1-forms $\omega_0^4 = -\langle d\mathbf{T}_0, \mathbf{T}_1 \rangle$ for any local section $T : U \subset \Lambda \rightarrow G$. Lie sphere geometry is the study of Legendre immersions $\lambda : M \rightarrow \Lambda$. Any immersion $f : M \rightarrow \mathcal{M}$ with tangent sphere map $S : M \rightarrow \mathbf{S}^{3,1}$ along it has a *Legendre lift*

$$(1) \quad \lambda = [f, S + \boldsymbol{\epsilon}_5] : M \rightarrow \Lambda.$$

This is a Legendre immersion, for if $F : U \subset M \rightarrow \mathbf{R}^{4,1}$ is any lift of f , then $\langle F, S + \boldsymbol{\epsilon}_5 \rangle = \langle F, S \rangle = 0$, so $[F, S + \boldsymbol{\epsilon}_5] \in \Lambda$, and the pull-back

by λ of the contact structure is $-\langle dF, S + \epsilon_5 \rangle = -\langle dF, S \rangle = 0$, all since S is a tangent sphere map along f . The spherical projection of the Legendre lift of f is again f . The spherical projection of a general Legendre immersion λ is a smooth map $\pi \circ \lambda : M \rightarrow \mathcal{M}$, but not necessarily an immersion.

Example 10. $\lambda : \mathbf{R}^2 \rightarrow \Lambda$, $\lambda(u, v) = [\mathbf{S}_0(u), \mathbf{S}_1(v)]$, where

$$\mathbf{S}_0(u) = \cos u \epsilon_0 + \sin u \epsilon_3 + \epsilon_4, \quad \mathbf{S}_1(v) = \cos v \epsilon_1 + \sin v \epsilon_2 + \epsilon_5.$$

is a smooth Legendre immersion. Its spherical projection $\sigma \circ \lambda : \mathbf{R}^2 \rightarrow \mathcal{M}$ is $\sigma \circ \lambda(u, v) = [\mathbf{S}_0(u)]$, which is f_+ of the great circle $\cos u \epsilon_0 + \sin u \epsilon_3$ in \mathbf{S}^3 . In particular, its spherical projection is singular at every point of $M = \mathbf{R}^2$.

Given a Legendre immersion $\lambda : M \rightarrow \Lambda$, for each $m \in M$, the line $\lambda(m)$ is the pencil of tangent spheres at m . A smooth map $S : U \subset M \rightarrow Q$ such that $S(m) \in \lambda(m)$, for each $m \in U$ is a *tangent sphere map* along λ . The definitions of curvature sphere map and Dupin Legendre immersions is the same as for immersions into \mathcal{M} . The important, but elementary, fact is that the Legendre lift (1) of a Dupin immersion $f : M \rightarrow \mathcal{M}$ with tangent sphere map $S : M \rightarrow \mathbf{S}^{3,1}$ is a Dupin Legendre immersion.

An elementary calculation verifies that the Legendre immersion of Example 10 is Dupin.

The best Lie frame field $T : U \rightarrow G$ along a Dupin Legendre immersion $\lambda : M \rightarrow \Lambda$ is characterized by $[\mathbf{T}_0] = S_0$ and $[\mathbf{T}_1] = S_1$ are the curvature spheres of λ and

$$\text{Order 1: } \omega_0^2 = 0 = \omega_1^3, \quad \theta^2 = \omega_1^2, \quad \theta^3 = \omega_0^3, \quad \theta^2 \wedge \theta^3 \neq 0,$$

$$\text{Order 2: } 0 = \omega_0^1 = \omega_1^0 = \omega_3^2 = -\omega_2^3,$$

$$\text{Order 3: } 0 = \omega_2^0 = \omega_3^1 = \omega_4^0.$$

By the structure equations of G ,

$$d\theta^2 = p\theta^2 \wedge \theta^3, \quad d\theta^3 = q\theta^2 \wedge \theta^3,$$

for smooth functions $p, q : U \rightarrow \mathbf{R}$. The remaining entries of ω are given by

$$\omega_0^0 = q\theta^2 + t\theta^3, \quad \omega_1^1 = u\theta^2 - p\theta^3, \quad \omega_3^0 = c_i\theta^i, \quad \omega_2^1 = d_i\theta^i,$$

for smooth functions $t, u, c_i, d_i : U \rightarrow \mathbf{R}$, where $i = 2, 3$. Taking the exterior differential of these forms, we get

$$\begin{aligned} dq \wedge \theta^2 + dt \wedge \theta^3 &= -(c_2 + q(p + t))\theta^2 \wedge \theta^3, \\ du \wedge \theta^2 - dp \wedge \theta^3 &= (d_3 + p(q - u))\theta^2 \wedge \theta^3. \end{aligned}$$

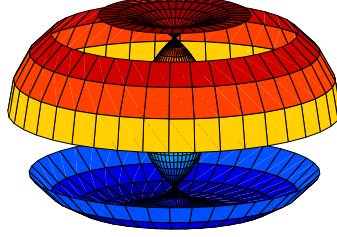


FIGURE 3. Spherical projection followed by hyperbolic stereographic projection of a right coset of H .

Lemma 11. *The left-invariant 6-plane distribution \mathcal{D}^\perp defined on G by*

$$\mathcal{D} = \{\omega_0^2, \omega_1^3, \omega_1^0, \omega_3^0, \omega_2^2, \omega_2^0, \omega_3^1, \omega_4^0, \omega_0^4\}$$

is completely integrable. Its maximal integrable manifolds are the right cosets of the connected 6-dimensional Lie subgroup H of G whose Lie algebra is

$$\mathfrak{h} = \mathcal{D}^\perp = \{\mathcal{T} \in \mathfrak{g} : \alpha(\mathcal{T}) = 0, \text{ for all } \alpha \in \mathcal{D}\}.$$

Theorem 12. *If $\lambda : M \rightarrow \Lambda$ is any Dupin Legendre immersion, then $\lambda(M)$ is an open submanifold of AHo , for some element $A \in G$.*

Proof. Each point of M has a neighborhood U on which there exists a best Lie frame field $T : U \rightarrow G$ along λ with $T^{-1}dT \in \mathfrak{h}$ on U , where $\mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra of H . Thus, $T : U \rightarrow G$ is an integral surface of the 6-plane distribution defined on G by \mathfrak{h} , so $T(U) \subset AH$, for some element $A \in G$, since the integral submanifolds of \mathfrak{h} are the right cosets of H . If M is connected, then $S(V) \subset AH$, for any best Lie frame $S : V \subset M \rightarrow G$. Hence, $\lambda(M) \subset AHo$. \square

Calculation of the best Lie sphere frame field along the Dupin Legendre immersion in Example 10 reveals that $\lambda(\mathbf{R}^2) = Ho$. Thus, the spherical projection $\sigma(Ho)$ is a great circle. For fixed $t \in \mathbf{R}$, if $A \in G$ is the matrix of

$$\begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_4 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \in \mathbf{SO}(4, 2),$$

in the Lie frame, then $\mathcal{S} \circ f_+^{-1} \circ \sigma(AHo)$, the stereographic projection of the spherical projection of AHo , is a surface of revolution obtained by rotating a circle about an axis that intersects it. It is singular at the points of intersection. Figure 3 shows it with part of the outer surface removed to reveal the inner part and the singularities.

In conclusion, every connected Dupin immersion in \mathbf{R}^3 is obtained in this way for some choice of element $A \in G$.

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